Strategic Decentralized Matching

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Outline

Propositions 1-3

- Albeit some important conceptual differences, **in the presence of complete information, centralized and decentralized markets have been shown to be very similar** in terms of the implementability of the stable match
- In fact, the stable match was implementable through DA heuristic mechanisms
- In addition, the equilibrium strategies which implemented the stable match were the **only ones which survive IEWDS**
- Finally, **time frictions did not really matter** in centralized markets, nor did they in decentralized markets with complete information

Deviations with Discounting

- **Equivalence** of centralized and decentralized markets **is broken once we combine** these two elements: **incomplete information and time frictions**
- **Two sources of deviation** or two hurdles to equilibrium implementation through DA: **Speeding-up** & **Belief Manipulation** (Why do we only care about deviation strategies to DA?)

Speeding-Up Example

- Nothing conceptually wrong with this example (in fact, same intuition uniquely identified the one-shot eqm under complete info)
- However DA-based strategies get derailed because of **time frictions** which **shift the determinant of offering order from** *preference* to *plausibility*
- So if we are really committed to the idea of implementing DA, we will need to leverage the uncertainty faced by firms to prevent speeding

$$U^f = U^w = \begin{array}{|c|c|}\hline 4 & 1 \\ \hline 3 & 2 \\ \hline \end{array}$$

- **Intuition:** We can cook up an example where **workers learn about the state through the timing of offers**. Eg. Under U_1 j expects to receive an offer from f_1 in $t = 1$, and to receive it in $t = 2$ under U_2. If f_1 finds it profitable to induce j to think they are in U_1, it will have incentives to deviate from DA in U_2

- Although authors make a fuss about this, there is **nothing deep about time**. Offer timing happens to be the only instrument available to firms in this context, hence learning (if any) should come through time

- We conjecture that similar results in spirit will hold under different instruments if these were available to the agents (eg. monetary contracts)

- In other words, the use of **time as an instrument is not related to exploiting time frictions**, but to the operating nature of the DA-mechanism

$$U_1 = \begin{array}{|c|c|c|}\hline \mathbf{5} & 4 & 3 \\ \hline 7 & \mathbf{10} & 2 \\ \hline 6 & 8 & \mathbf{9} \\ \hline \end{array} \qquad U_2 = \begin{array}{|c|c|c|}\hline 5 & 9 & \mathbf{4} \\ \hline 7 & \mathbf{10} & 3 \\ \hline \mathbf{6} & 8 & 2 \\ \hline \end{array} \qquad U_3 = \begin{array}{|c|c|c|}\hline \mathbf{5} & 4 & 3 \\ \hline 2 & 6 & \mathbf{9} \\ \hline 1 & \mathbf{8} & 7 \\ \hline \end{array} \qquad U_4 = \begin{array}{|c|c|c|}\hline \mathbf{5} & 4 & 3 \\ \hline 9 & 7 & \mathbf{10} \\ \hline 6 & \mathbf{8} & 2 \\ \hline \end{array}$$

- **Spoiler:** f_1 finds beneficial to make an early offer in $t = 1$ to w_1 in U_2 exploiting the fact that under DA, a $t = 1$ offer is to be expected only in U_1

- What's the logic? First we see that in U_3, f_1 makes an offer to w_1 at some point (because workers know exactly where they are, so for δ high enough, they know that by waiting they can just do better)

- In U_4, f_3 makes an offer to w_2 straightaway (despite not knowing $U = U_2 \; OR \; U_4$. Payoffs from w_1 are $(6, 6\delta)$, while payoffs from w_2 are $(8, 6\delta)$

- **Tricky part:** In U_2 we know that w_2 and f_2 match in $t = 1$. Moreover, we've just seen how f_3 also makes an offer to w_2 which is rejected and then it makes an offer to w_1 in $t = 2$. Certainly, any stable eqm should require f_1 to make an offer to w_3 in $t = 1$ (which will be accepted straightaway)

- But is this true? Can f_1 get any better? YES! **f_1 can leverage on w_1 incompleteness to make an offer in $t = 1$.** This offer is accepted because w_1 doesn't know if $U = U_1 \; OR \; U_2$. In U_1 it is optimal for f_1 to make an offer in $t = 1$ to w_1, and for w_1 to accept it provided that no further offers will arrive. Hence accepting an offer in $t = 1$ coming from f_1 is undominated for w_1. This leads to an unstable eqm (which would survive even IEWDS!)

- What has just happened? f_1 leveraged the gap in information of w_1 to make an offer which is compatible with w_1 best strategy
- Authors suggest that this phenomenon was due to the information transmission via timing (which is true by design), but it looks like an empty statement to me. Anyway...
- **How do they solve this problem?** By brute force: they manually close the door. How? **Creating a new market where waiting is optimal, so waiting is undominated**
- If a new market where it were possible to get a higher payoff by waiting existed (and δ is high), then why accepting f_1 straightaway? Simply hold f_1 offer for one period, and accept it conditional on no more offers arriving in $t = 2$. But waiting is enough to break f_1's deviating strategy!

Closing the Door through Economy Richness

- So **intuitively full market support will close all the doors**. If there are infinitely many potentials, (i) in some of them it would be optimal to always wait a bit more to avoid these kamikaze strategies and (ii) in some of them speeding would be too risky (you might be losing on sth by not waiting)
- The **difficult part is to ensure that such potentials can indeed be built while being compatible with the observed history** of events
- As it happens, **full support is** a bit of an **overkill**. Intuitively **we just need** that for every agent a market exists where it is **too risky to speed up**, and where it is **always worth it to wait** a bit more to avoid an early matching
- These two conditions are summarised in **Assumption 1** and **Assumption 2**, and we refer to them as **Generalized Richness**

Structure of the Rest of the Session

- Present DA-rules in this context. Why so stubborn with DA? [**Lemma 2**]
- Introduce formally Assumption 1 and Assumption 2
- Show that Assumption 1 and Assumption 2 are sufficient for the implementability of stable outcomes via DA strategies (it's messy but kind of obvious, as they are devices specifically designed to avoid deviations in DA-based strategies) [**Proposition** 4^*]
- Show that full-support (G) implies A1 and A2 (G^*) (ie. that we can construct markets satisfying A1 and A2 while being history compatible)
- Prove Claim 1 as intermediate step in $G \implies G^*$

Proposition 4

Proposition 4: *Suppose the economy is rich (has full-support). In the decentralized market game with sufficiently high δ, there exists a Bayesian Nash equilibrium in weakly undominated strategies that implements the unique stable matching in each supported market.*

Disclaimer: Such equilibrium is **not unique and minor refinements won't get rid off unstable eqm** like IEWDS did so far.

Why so stubborn with DA?

- Start by looking at the features that every strategy aiming for stability should satisfy in the centralized game (ie. **ex-ante stability conditions**)
- Strategies refer to mappings from information (utilities) to preferences (that are revealed to the authority)
- Define $P(v)$ and $A(v)$ (confusingly called the acceptable set, because interimly you'd accept any offer). Let $S_l^\alpha(v = u_l^\alpha)$ be the plausible set
- **Definition:** Submitting v (equivalent to submit $P(v)$, $A(v)$) is a DA-strat if
  - $S_l^\alpha(v) \subseteq A(v)$
  - For all $k, m \in S_l^\alpha(u_l^\alpha), kP(v)m \iff u_l^\alpha(k) > u_l^\alpha(m)$
  - For all $k \in A(v) \backslash S_l^\alpha(u_l^\alpha)$, $m \in S_l^\alpha(u_l^\alpha)$, $kP(v)m \implies u_l^\alpha(k) > u_l^\alpha(m)$

- This DA-strat entails some **associated behaviour**. Notably, **firms can only make offers to workers as good as their preferred stable match**
- **Workers can only reject offers from firms worse than their least preferred stable match**
- What it doesn't say? It does **not impose any restrictions on how agents should behave wrt agents out of the stable set**. It also doesn't preclude agents from ranking a plausible partner over a non-plausible partner, even if the utility under the latter is higher
- **Definition:** A rule is a strategy which only relies on the set of participants, the realized agent's utilities and S

Why did we focus on DA and its deviations? Because **DA are almost iff for stability** in the centralized game

Lemma 2

1. *If all agents use a reduced DA rule then [...] the outcome produced by the DA mechanism is the unique stable matching*
2. *All agents using reduced DA constitute a BNE*
3. *If any agent deviates from DA, then this corrupted DA mechanism may yield an unstable outcome*

Implication: DA are not only very good at implementing stable outcomes (we kind of knew that already), but they are (almost) necessary. Deviations from DA lead to instability.

Proof of Lemma 2

- *If all agents use a reduced DA rule then [...] the outcome produced by the DA mechanism is the unique stable matching*

- **Proof:** Suffices to show that DA yields μ_M which is the unique stable match by Propositon 1. Assume for contradiction a DA yields a match $\mu \neq \mu_M$. By Prop 1, μ is unstable hence a blocking pair (i', j') must exist st $\mu_M(i') = j'$, namely $u_{i',j'} \geq u_{i',\mu(i')}$ and $u_{i',j'} \geq u_{\mu(j'),j'}$. From $\mu_M(i') = j'$ we know that i' and j' are in each other's stable set, hence in case of receiving an offer from i', j' could have not possibly rejected i' (as $i' \in S_{j'}$ and $u_{i',j'} > u_{\mu(j'),j'}$). But a similar argument shows that j' must have asked i' before asking $\mu(i')$. So $\mu(j') = i'$. Contradiction.

- *All agents using reduced DA constitute a BNE*
- **Proof:** The clearinghouse emulates a firm-proposing DA which is SP, so truthful revelation is weakly dominant for firms.
- For workers strictly profitable deviations imply that $\mu_{\sigma_j}(j) \neq \mu_M(j) : u_{\mu(j),j} > u_{\mu_M(j),j}$. From Part 1, a blocking pair (i', j') st $\mu_M(i') = j'$ must exist. Fair enough $j' \neq j$ because $u_{\mu(j),j} > u_{\mu_M(j),j}$ it's not blocking. Hence both i' and j' must be playing DA. But if both are playing DA (and i' prefers j' to $\mu(i')$ and $j' \in S_{i'}$), i' must have asked j' before $\mu(i')$. But j' could have not possibly rejected this offer (same argument as Part 1). Hence they cannot be a blocking pair. Contradiction

- *If any agent deviates from DA, then this corrupted DA mechanism may yield an unstable outcome*

- **Sketch of a Proof:** There are 3 ways an agent can break a DA rule,
  - By declaring a plausible (stable) match unacceptable. But then, by definition of the stable set, some outcomes will be unstable.
  - By ranking a plausible match i below an agent $i' \notin S, u_{\cdot,i} > u_{\cdot,i'}$. They build an example to show how this can lead to unstable eqa. Consider a worker j, misranking i and i'. The idea is to design a game where i' asks j (following DA prescriptions, this means, without knowing that j will deviate), and j accepts such offer according to its deviation, leading to an unstable eqm.
  - Same heuristics hold for misranking within the plausible set, and for firms $\square$

Implications of Interim DA-strat?

- What additional behaviour does it entail for agents to follow DA interimly?
- **Firms** - Easy: (If not matched, or offer is not currently held) Propose to most preferred worker (who has not rejected you yet, according to v). When last acceptable worker exits the market you leave
- **Workers** - Harder: (i) Accepting offers only coming from firms as good as most preferred (not yet exited) firms, (ii) rejecting all offers from firms less preferred than the least preferred (still available) plausible firm

- For the rest of the analysis we'll be looking at an even smaller set of (interim) DA strategies: *minimal* DA. **A decentralized strategy is minimal if** (i) firms only ask potential stable matches $A(v) = S_i^f(u_i)$ and (ii) if workers $\forall k \in S_j^w(u_j), u_j^w(k) > u_j^w(i) \implies i \notin A(v)$
- This implies that firms only need to rank plausible matches and workers only rank firms at least as good as the least preferred potential stable match
- **Proposition** 4* **precisely shows how A1 and A2 remove the incentives to deviate from interim DA**. The intuition (once again) is that for δ high enough (and because no-cycle property of aligned preferences), there exists a market where any stable match can be realized. The cumulative loss associated to such deviation can deter agents from deviating.

- **Assumption 1:** *Suppose all agents use minimal DA. For every firm i, market realization U, and period t, it must hold that for all w_j ranked below the most preferred worker in S (that has not rejected the firm yet), either*
  - $\exists\, \tilde{U} \in G(u_i^f, h_{t,i}^f) : \tilde{u}_{ij}^w > \tilde{u}_{\mu_M(\tilde{U},j)j'}$, or
  - for all $\tilde{U} \in G(u_i^f, h_{t,i}^f)$, $\tilde{u}_{ij}^w < \min\{\tilde{u}_{kj}^w : k \in S_j^w\}$
- In words, there **always exist a market where the worker you are deviating to prefers you to its stable match** (so it could accept your offer straightaway, making you potentially worse off as, by the definition of the plausible set, you could have actually obtained your preferred stable match)
- OR that every worker you deviate you to considers you unacceptable, so you are rejected immediately (hence if anything, you've only delayed your match with your preferred plausible match)

Assumption 1 and Assumption 2 - Generalized Richness

- **Assumption 2:** *Suppose all agents follow minimal DA. Define BS_t as the "mechanically" updated S_0 for firms, and the "mechanically" updated S_0 which have not asked j yet and are preferred to current offers for workers. Consider $j, k \in BS_t$, then if $u_l^{\alpha}(j) > u_l^{\alpha}(k)$ and $k \in$ updated S, so must be j*

- In words, think of BS as default plausible sets. Regardless your update of BS (given a history sequence), and regardless the ranking within BS, if two agents are in BS, they cannot be dropped from S. For instance, a worker seeing a firm in his default set leave, cannot make him drop any of the other firms out of S_t. More generally, **A2 says that the timing of other matches or offers cannot have an impact on** S (they should not carry "any information"). There is an exception when a worker receives an offer from every firm but one

- An economy satisfies **generalized richness** G^* if it satisfies A1 and A2

Proposition 4* *For sufficiently high δ, generalized richness implies the existence of a BNE in WUS of the decentralized game that implements the unique stable matching*

Intuition: We had two sources of deviation in decentralized (minimal) DA: Speeding-up and Belief Manipulation. We can “simply” create markets that make Speeding-up and Belief Manipulation risky (for every history sequence). High δ makes any non-zero risk bite.

Proof Proposition 4*

- For workers. Show (any) minimal DA is optimal when all other workers and firms use minimal DA. DA **for workers** implied
- (i) Accepting offers only coming from firms as good as most preferred firms, (ii) Rejecting all offers from firms less preferred that the least preferred plausible firm
- It follows directly (without invoking G^*) that for high enough δ exiting the market or accepting an offer from not the most preferred available firm in S cannot be optimal. Also, there's no benefit from holding offers other than its most preferred (by the non-cycle property, and given that rejected firms do not ask again under DA)

- **For firms**. Show the unique minimal DA is optimal for firms when all other firms and workers use minimal DA. Remember DA for firms implied:
- Propose to most preferred worker, who has not rejected you yet. When last acceptable worker exit the market you leave. Hence two possible deviations:
- (i) **Making no offer** (what may impact workers' history). By A2, such impact cannot exclude any firm from S (inc i), so at best it's just delaying the match
- (ii) **Not making an offer to** $j^* \in S$. A1 ensures that either i is rejected (hence delaying the stable match) or it's actually accepted (making i worse-off bc by definition of $j^* \in S$, a market where $\mu(i) = j^*$ exists). $\square$

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- Remains to show that $G \implies G^*$. Because G includes every possible market realization, it suffices to show that we can actually build **history consistent** ordinal potentials while accommodating A1 and A2
- **Prove constructively that such potentials can be built** (first from a firm pov, then worker)
- Assume wlog that utilities take values in $\{\varepsilon, 1, \ldots, n\}$ and $\Phi_{ij} \in \mathbb{Z}_0^-$

- Pick any i and start by making Φ **consistent with the public history**:
- For any i^1, j^1 that matched in period 1 set $\Phi_{i^1 j^1} = 0$, for any i^2, j^2 which exit in period 2, pick a random firm i^1 (with the set non-empty by top-top) and set $\Phi_{i^1 j^2} = -j$ and $\Phi_{i^2 j^2} = -(n+1)$ (this ensures that even if firm i^2 had asked j^2 to leave in $t = 1$, he would have held the offer waiting for i^1 to make an offer in the future, and after seeing it leaving it accepts i^2)
- Moreover, we need to ensure that i^2 in fact asks j^2 (in $t \leq 2$) and j^2 accepts. Hence, both the i^2 row and the j^2 column must be lower than $-(n+1)$.
- Iterate this process for every pair i^k, j^k such that $\Phi_{i^k, j^k} = -(k-1)(n+1)$ and $\Phi_{i^{(k-1)}, j^k} = -(k-2)(n+1)$. This restrictions rationalizes all public info

- Rationalizing **firm's private history** h_t (own preference ranking, offers it made and their status). Knowledge of own preference entails knowledge of row i (which has been only partially limited by our construction above)

- Now, can Φ explain i's offer-related patterns? **If firm cannot move** in t (ie. it matched, exited or is in hold) **no deviation is possible** so Prop 4 holds

- We can then **focus on cases where all i's offers have been rejected by** t. Combinations of these two sequences might have taken place up to t:

- i made an offer to j, who held it for some number of periods ≥ 0, then rejected it and left in $t' \leq t$ with firm i'. Based on the construction above $\Phi_{i'j} = -(t'-1)(n+1)$. Moreover $\Phi_{i'j'} \geq \Phi_{i'j}$ for all j' who exited before t' and $\Phi_{i',j} \geq \Phi_{i'j'}$ for all j' who remained after j. This ensures that the offer arrives in t'. Nothing in h_t^i informs the preferences of j wrt to remaining firms, so $\Phi_{i''j} \leq \Phi_{i'j}$ is consistent

- Similar intuition holds for the event where firm i makes an offer, j holds it for some periods ≥ 0, and in $t' \leq t$, j rejects i, but he doesn't exit the market. Construction follows the same pattern as above with at least one $i'' : \Phi_{i'',j} \geq \Phi_{i',j}$. We can rationalize worker j waiting any time length $t - t' < n - t'$ without leaving
- Finally, **to rationalize the order of offers** of firm i with Φ_i, simply set $\Phi_{ij} \leq -t(n+1)$ for all j who've not been asked by t (which is true in Φ)
- Finally, observe that Φ ensures that the workers matched in t according to the protocol above, are still around
- We've just shown that any history of events which an arbitrary firm might have been exposed to, can be rationalized by a Φ of the form described above

- It **remains to prove that A1 and A2 are also compatible** with such Φ

- Proof is actually quite simple. For A1, simply consider that for any j which is less preferred than most preferred worker (and which consequently has not been asked by period t) has $\Phi_{ij} = -n(n+1) - j$ and $\Phi_{i'j} = -n(n+1) - j - i'$. The **restrictions above only constrained rows and columns of matched agents** (not in S_t), the **row of** i and the **columns of** j_t^*, hence this restriction can be immediately satisfied

- A2 (for firms) implied not dropping anyone from S_t except under exiting. **Suffices to show that every remaining worker can be a top match**. To do so, set $\Phi_{ij^*} = -(t-1)(n+1)$. Every other match so far could have had $\Phi_{ij} > -(t-1)(n+1)$, hence $\Phi_{ij'} \in -(t-1)(n+1) - j'$ is consistent and it would made j' accepting the offer straightaway

- **Worker's history includes own preferences, sequence of offers up to** t **and their status**. Intuition is to build "optimistic potentials" to make every worker believe that $i^* \in S$. Start by introducing this intermediate result
- **Claim 1:** *Suppose all agents follow reduced DA. If j rejects i in t and in $t' > t$, i exits while j remains in it, then there must exist an i' that has not made an offer to j also exiting in t'*
- **Proof:** Firm i only makes an offer to j if $j = j^* \in S_i$. Hence $j' \in S : u_{ij} > u_{i\mu(i)}$, namely i cannot be leaving with anyone better than j (after being rejected by j). But since j is still in the game, i is not leaving as a top-top match (which must be formed in every period!). $\square$

- Imagine i and j' exited in t. If i has made an offer to j then we set $\Phi_{ij'} = -(n+1)^2$ (not top-top match), and if not $\Phi_{ij'} = -(t-1)(n+1)$ (as in public info setting)

- For all firms who made no offers to j in period 1 (and haven't exited), pick a j^1 who matched in $t=1$, and set $\Phi_{ij^1} = -i$ (ie. they all asked j^1 out, and all but one were rejected). Moreover $-i > -(n+1)$ which is the Φ_{ij} of top-top match in $t=2$, so none of them would have asked anybody else but j^1 (not even j surely)

- In $t=2$, if j rejected any offers in $t=1$, there must be a worker j^2 who exited with a firm who did not make an offer to j in $t=1$ (by Claim 1). In case no offers (or holding offers), such j^2 exists by top-top. For all firms who did not make offers to j up to period t, simply set $\Phi_{ij^2} = -(n+1)-i$

- Repeat this construction for all $j^{t'}$, $\Phi_{ij^{t'}} = -(t'-1)(n+1)-i$

- Now for firms which did make offers to j. If one of them is the best remaining firm, DA prescribes to take it (G^* does not bite). Otherwise, DA prescribes to keep the best one (i_j^1), and drop all the rest. Simply assign $\Phi_{i_j^k j} = -(n+1)^2 + u^w_{i_j^k j}$, $\Phi_{i_j^1 j} = -t(n+1)$ and $\Phi_{i_j^* j} = -(t-1)(n+1) - i - 1$.
- Finally, need to show that A1 and A2 hold under these restrictions. A1 holds immediately, bc it only restricts firms beliefs
- For A2. If only one remaining firm, A2 is satisfied automatically. If j's received an offer from most preferred firm (there is no further remaining preferred firm so A2 also holds). Finally, assume worker j received one offer not coming from its most preferred firm. Then by the construction above we can set $\Phi_{ij} = -(n+1)^2 + u^w_{ij}$, and $\Phi_{i^* j} = -(t-1)(n+1) - i$ (what explains why they have not asked j yet, and why it would be optimal to wait one more period as i^* would form a top-top match with j) $\square$